A Compositional Sheaf-Theoretic Framework for Event-Based Systems

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7th July, 2020

Engineering complex systems causes pain

	actuation	perception	hardware
	sensing	planning	localization
robot design =	control	learning	mapping
	energetics	interactions	coordination
	computation	software	calibration

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$$\text{robot design} = \begin{matrix} \text{actuation} & \text{perception} & \text{hardware} \\ \text{sensing} & \text{planning} & \text{localization} \\ \text{control} & \text{learning} & \text{mapping} \\ \text{energetics} & \text{interactions} & \text{coordination} \\ \text{computation} & \text{software} & \text{calibration} \end{matrix}$$

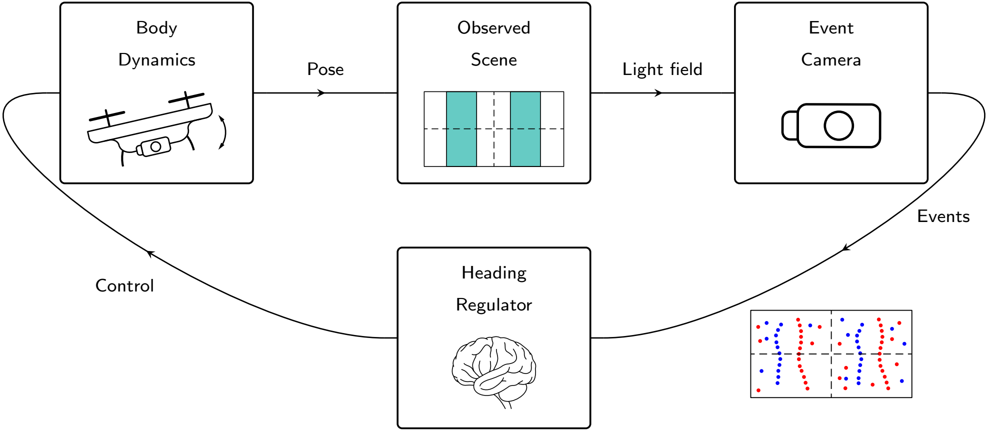
Engineering malaise:

- Too many components, too different (continuous, discrete, event-based).
- Each component is modeled in a specific way: Engineers like to partition.
- Even if modeled differently, the components have to be co-designed at some point: Pain.

An example: Event-Based Sensors

`https://www.youtube.com/watch?v=kPCZESVfHoQ`

How can we put together continuous time, event-based, discrete time?



Towards a solution: A common framework

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- Components are different, but there is an abstraction level at which they are the same.
- These are *behavior types*, which can be *composed* in different ways.

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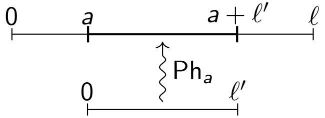
Some references:

- Schultz, Spivak, Vasilakopoulou, *Dynamical systems and sheaves*, '20.
- Schultz, Spivak, *Temporal Type Theory: A topos-theoretic approach to systems and behavior*, '19.

Describing system behaviors: The category of continuous intervals

Given $a \in \mathbb{R}_{\geq 0}$, the translation-by-$a$ function is

$$\begin{aligned} \mathrm{Ph}_a \colon \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ \ell &\mapsto a + \ell \end{aligned}$$



Definition (Category of continuous intervals Int)

- Objects: *Durations* $\mathrm{Ob}(\mathrm{Int}) := \{\ell \in \mathbb{R}_{\geq 0}\}$.
- Morphisms: Given durations ℓ', ℓ, one has $\mathrm{Int}(\ell', \ell) := \{\mathrm{Ph}_a \mid a \in \mathbb{R}_{\geq 0} \text{ and } a + \ell' \leq \ell\}$.
- Identity morphism: $\mathrm{id}_\ell := \mathrm{Ph}_0 \in \mathrm{Int}(\ell, \ell)$.
- Composition of morphisms: Given $\mathrm{Ph}_a \colon \ell \to \ell'$, $\mathrm{Ph}_b \colon \ell' \to \ell''$, one has

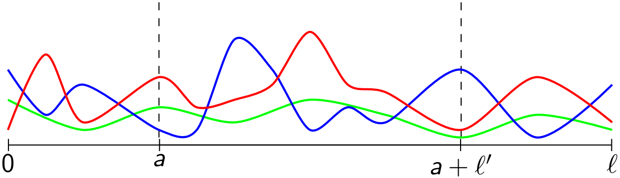
$$\mathrm{Ph}_a \fatsemi \mathrm{Ph}_b = \mathrm{Ph}_{a+b} \in \mathrm{Int}(\ell, \ell'').$$

Describing system behaviors: Int-presehaves and sections

Definition (Int-presheaves)

An Int-*presheaf* A is a functor $A\colon \mathrm{Int}^{\mathrm{op}} \to \mathrm{Set}$.

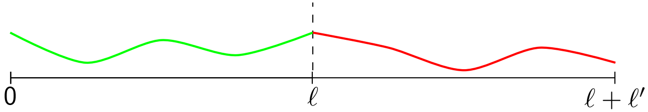
- $x \in A(\ell)$ is a *length-ℓ section (behavior)*.
- Given $x \in A(\ell)$, $\mathrm{Ph}_a\colon \ell' \to \ell$, the *restriction* is $x|_{[a,a+\ell']} = A(\mathrm{Ph}_a)(x) \in A(\ell')$.



Describing system behaviors: Compatible sections and Int-sheaves

Definition (Compatible sections)

Given a presheaf A, the sections $a \in A(\ell)$ and $a' \in A(\ell')$ are *compatible* if $a|_{[\ell,\ell]} = a'|_{[0,0]}$.



Describing system behaviors: Compatible sections and Int-sheaves

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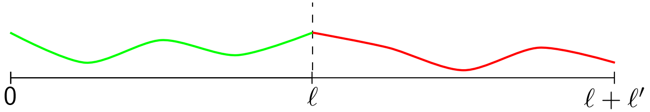
Definition (Int-sheaf)

An Int-presheaf $A\colon \mathsf{Int}^{\mathrm{op}} \to \mathsf{Set}$ is an Int-*sheaf* if, for all ℓ, ℓ' and compatible sections $a \in A(\ell)$, $a' \in A(\ell')$, there exists a unique $\bar{a} \in A(\ell + \ell')$ such that

$$\bar{a}|_{[0,\ell]} = a \quad \text{and} \quad \bar{a}|_{[\ell,\ell+\ell']} = a'.$$

We denote by $\widetilde{\mathsf{Int}}$ the category of Int-sheaves.



Describing different components: Machines

Definition (Machine)

Let $A, B \in \widetilde{\mathrm{Int}}$. An (A, B) *machine* is a span in $\widetilde{\mathrm{Int}}$ of the form:

$$\begin{tikzcd} & C \arrow[dl, "f^{\mathrm{in}}"'] \arrow[dr, "f^{\mathrm{out}}"] & \\ A & & B \end{tikzcd}$$

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Example (Continuous Dynamical System)

Consider A, B euclidean spaces. An (A, B)-continuous dynamical system consists of

- State space: S.
- Dynamics: $\dot{s} = f^{\mathrm{dyn}}(s, a)$, for $a \in A$, $s \in S$, and smooth f^{dyn}.
- Readout: $b = f^{\mathrm{rdt}}(s)$, $b \in B$ and smooth f^{rdt}.

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$$\begin{tikzcd} & D \arrow[dl, "f^{\mathrm{in}}"'] \arrow[dr, "f^{\mathrm{out}}"] & \\ C^1(A) & & C^1(B) \end{tikzcd}$$

$$D(\ell) = \{(a, s, b) \in C^1(A) \times S \times C^1(B) \mid \dot{s} = f^{\mathrm{dyn}}(a, s),\ b = f^{\mathrm{rdt}}(s)\}$$

Composing components via machines composition

Definition (Composition of machines)

Given two machines $M_1 = (D_1, f_1^{\text{in}}, f_1^{\text{out}})$ and $M_2 = (D_2, f_2^{\text{in}}, f_2^{\text{out}})$ of types (A, B) and (B, C) respectively, their *composite* is the machine $M = (D_1 \times_B D_2, f^{\text{in}}, f^{\text{out}})$ of type (A, C), namely the span given by pullback:

$$
\begin{tikzcd}
 & & D_1 \times_B D_2 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\
 & D_1 \arrow[dl, "f_1^{\text{in}}"'] \arrow[dr, "f_1^{\text{out}}"] & & D_2 \arrow[dl, "f_2^{\text{in}}"'] \arrow[dr, "f_2^{\text{out}}"] & \\
A & & B & & C
\end{tikzcd}
$$

A closer look to a particular behavior type: Event streams

Definition (Event stream)

Given a set A and $\ell \geq 0$. A length-ℓ *event stream* of type A is an element of

$$\mathrm{Ev}_A(\ell) := \{(S, a) \mid S \subseteq \tilde{\ell}, S \text{ finite}, a\colon S \to A\}.$$

Example (Swiss traffic light)

$(S, a) \in \mathrm{Ev}_A(60)$, $S = \{20, 25, 45, 50\}$

$$a\colon S \to A = \{\mathrm{redToOrange}, \mathrm{orangeToGreen}, \mathrm{greenToOrange}, \mathrm{orangeToRed}\}$$

$$s \mapsto \begin{cases} \mathrm{redToOrange}, & \text{if } s = 20, \\ \mathrm{orangeToGreen}, & \text{if } s = 25, \\ \mathrm{greenToOrange}, & \text{if } s = 45, \\ \mathrm{orangeToRed}, & \text{if } s = 50. \end{cases}$$

Facts about event streams

Proposition (Ev is functorial)

Ev is functorial: Given a map $f\colon A \to B$, there is an induced morphism $Ev_f\colon Ev_A \to Ev_B$ in $\widetilde{\mathsf{Int}}$, preserving identities and composition.

Proposition (Ev_A is a sheaf)

For any set A, Ev_A is an Int*-sheaf.*

Proposition (Ev is a strong monoidal functor)

$Ev\colon (\mathsf{Set}, \odot, \varnothing) \to (\widetilde{\mathsf{Int}}, \times, 1)$ *is a strong monoidal functor, i.e.*

$$1 \cong Ev_{\varnothing} \text{ and } Ev_A \times Ev_B \cong Ev_{A \odot B},$$

where $A \odot B := A + B + A \times B$.

We are ready to define event-based systems: A particular type of machines

Definition (Event-based system)

Let A, B be sets. An *event-based system* $P = (C, f^{\text{in}}, f^{\text{out}})$ of type (A, B) is a machine

$$\begin{tikzcd} & C \arrow[dl, "f^{\text{in}}"'] \arrow[dr, "f^{\text{out}}"] & \\ \mathrm{Ev}_A & & \mathrm{Ev}_B \end{tikzcd}$$

between two event streams. Graphically:

$$\mathrm{Ev}_A - \boxed{\;P\;} - \mathrm{Ev}_B$$

Discrete dynamical systems as an example of event-based systems

Example (Discrete Dynamical Systems (DDS))

Let A, B be sets. An (A, B)-DDS consists of:

- State space S.
- Update function: $f^{\text{upd}} \colon A \times S \to S$.
- Readout function $f^{\text{rdt}} \colon S \to B$.

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Let A, B be sets. An (A, B)-DDS consists of:

- State space S.
- Update function: $f^{\text{upd}} \colon A \times S \to S$.
- Readout function $f^{\text{rdt}} \colon S \to B$.

This is an (A, B)-event-based system with

$$D(\ell) := \{T \subseteq \tilde{\ell}, (a, s) \colon T \to A \times S \mid T \text{ finite and } s_{i+1} = f^{\text{upd}}(a_i, s_i) \text{ for all } 1 \leq i \leq n-1\}.$$

Given $(T, a, s) \in D(\ell)$:

$$f^{\text{in}}(T, a, s) = (T, a),$$
$$f^{\text{out}}(T, a, s) = (T, (s \mathbin{⨾} f^{\text{rdt}})).$$

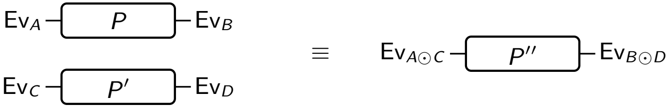
Tensor product of event-based systems

Definition (Tensor product of event-based systems)

Given two event-based systems $P(E, f^{\text{in}}, f^{\text{out}})$, $P'(E', f^{\text{in}\prime}, f^{\text{out}\prime})$, of types (A, B) and (A', B'), their tensor product is of type $(A \odot A', B \odot B')$ and is given by:

$$\begin{tikzcd} & E \times E' \arrow[dl, "f^{\text{in}} \times f^{\text{in}\prime}"'] \arrow[dr, "f^{\text{out}} \times f^{\text{out}\prime}"] & \\ \mathrm{Ev}_{A \odot A'} & & \mathrm{Ev}_{B \odot B'} \end{tikzcd}$$

Graphically:



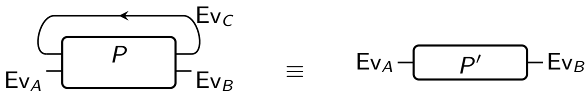
Trace of event-based systems

Definition (Trace of event-based systems)

Given an event based system $P(D, f^{\text{in}}, f^{\text{out}})$ of type $(A \times C, B \times C)$, its trace is an event-based system $P(E, f^{\text{in}}_{\text{tr}}, f^{\text{out}}_{\text{tr}})$ of type (A, B), with

$$
\begin{aligned}
E(\ell) &= \{d \in D(\ell) \mid \pi_2(f^{\text{in}}(d)) = \pi_2(f^{\text{out}}(d))\} \\
f^{\text{in}}_{\text{tr}}(d) &= \pi_1(f^{\text{in}}(d)) \in \text{Ev}_A \\
f^{\text{out}}_{\text{tr}}(d) &= \pi_1(f^{\text{out}}(d)) \in \text{Ev}_B.
\end{aligned}
$$

Graphically:



Yes, but what if I have to model continuous signals? Continuous streams

Definition (Continuous stream)

Given a topological space A, we can define the sheaf of *continuous streams* of type A as

$$\mathrm{Cnt}_A(\ell) := \{a \mid a\colon \tilde{\ell} \to A \text{ continuous}\}.$$

Lipschitz continuous streams are given by the sheaf:

$$\mathrm{LCnt}_A(\ell) = \{a\colon \tilde{\ell} \to A \mid a \text{ Lipschitz continuous}\} \subseteq \mathrm{Cnt}_A(\ell).$$

How to go from continuous to discrete? A sampler

Definition (Sampler)

Let A be a topological space and $d \in \mathbb{R}_{\geq 0}$ the *sampling time*. A *period-d A-sampler* is a span

$$\begin{tikzcd}
& \mathsf{Clock}_d \times \mathsf{Cnt}_A \arrow[dl, "f^{\mathrm{cnt}}"'] \arrow[dr, "f^{\mathrm{evt}}"] & \\
\mathsf{Cnt}_A & & \mathsf{Ev}_A
\end{tikzcd}$$

where:

$$\begin{aligned}
f^{\mathrm{cnt}} \colon \mathsf{Clock}_d \times \mathsf{Cnt}_A &\to \mathsf{Cnt}_A \\
(\phi, a) &\mapsto a, \\
f^{\mathrm{evt}} \colon \mathsf{Clock}_d \times \mathsf{Cnt}_A &\to \mathsf{Ev}_A \\
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... but does this capture all engineering applications?

Solution: A level crossing sampler

Definition (L-level-crossing sampler)

Let (A, dist) be a metric space and consider a Lipschitz input stream $\mathrm{LCnt}_A(\ell)$. Consider the *level* $L \in \mathbb{R}$. A *L-level-crossing sampler* of type A is a span with $P(\ell) := \{(c, a_0) \mid c \in \mathrm{LCnt}_A(\ell), a_0 \in A\}$:

$$\begin{tikzcd} \mathrm{LCnt}_A & P \arrow[l, "f^{\mathrm{cnt}}"'] \arrow[r, "f^{\mathrm{evt}}"] & \mathrm{Ev}_A \end{tikzcd}$$

- $f^{\mathrm{cnt}}(c, a_0) = c$.
- If $\mathrm{dist}(c(t), a_0) < L$ for all $t \in \tilde{\ell}$: $f^{\mathrm{evt}}(c) = (\varnothing, !)$.
- If there exists $t \in \tilde{\ell}$ with $\mathrm{dist}(c(t_1), a_0) \geq L$: $t_1 = \inf\{t \in \tilde{\ell} \mid \mathrm{dist}(c(t_1), a_0) \geq L\}$, $a_1 = c(t_1)$.
- Recursively, define $t_{i+1} \in \tilde{\ell}$ to be the least time such that $\mathrm{dist}(c(t_{i+1}), a_i) \geq L$ (if there is one).
- $f^{\mathrm{evt}}: P(\ell) \to \mathrm{Ev}_A, \quad (c, a_0) \mapsto \{t_1, \dots, t_n, c(t_1), \dots, c(t_n)\}$

How to go from discrete to continuous? A reconstructor

Definition (Reconstructor)

Let A be a set. A *reconstructor* of type A is a span

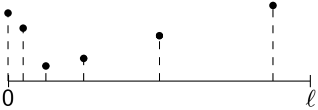
$$\begin{tikzcd} & C \arrow[dl, "f^{\mathrm{evt}}"'] \arrow[dr, "f^{\mathrm{cnt}}"] & \\ \mathrm{Ev}_A & & \mathrm{Cnt}_A \end{tikzcd}$$

with

$$C(\ell) = \{(S, a_0, a) \mid S \subseteq \tilde{\ell} \text{ finite}, a_0 \in A, a\colon S \to A\},$$

where $f^{\mathrm{evt}}(S, a_0, a) \coloneqq (S, a)$, and where $a' \coloneqq f^{\mathrm{cnt}}(a_0, s_1, \dots, s_n, a_1, \dots, a_n)\colon \tilde{\ell} \to A$ is given by

$$a'(t) \coloneqq \begin{cases} a_0, & 0 \leq t < s_1 \\ a(s_i), & s_i \leq t < s_{i+1}, \quad i \in \{2, \dots, n-1\} \\ a(s_n), & s_n \leq t \leq \ell. \end{cases}$$



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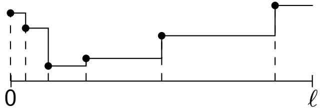
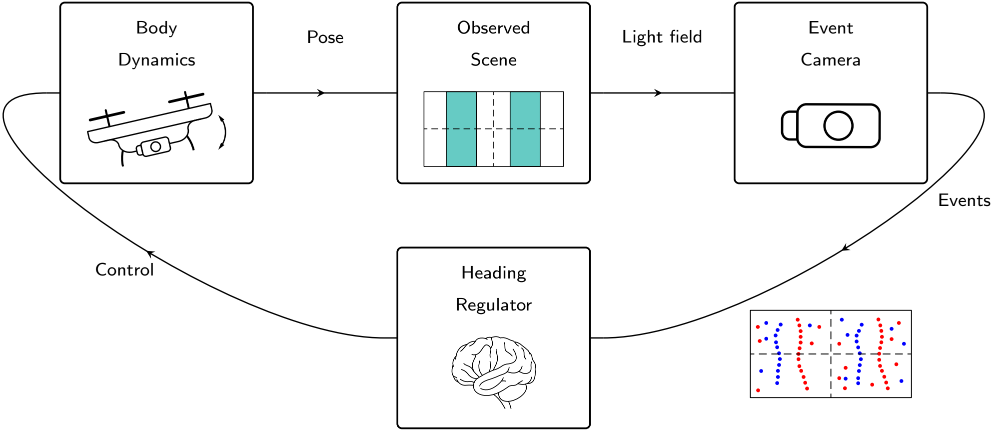


Neuromorphic heading regulation problem within a unified framework



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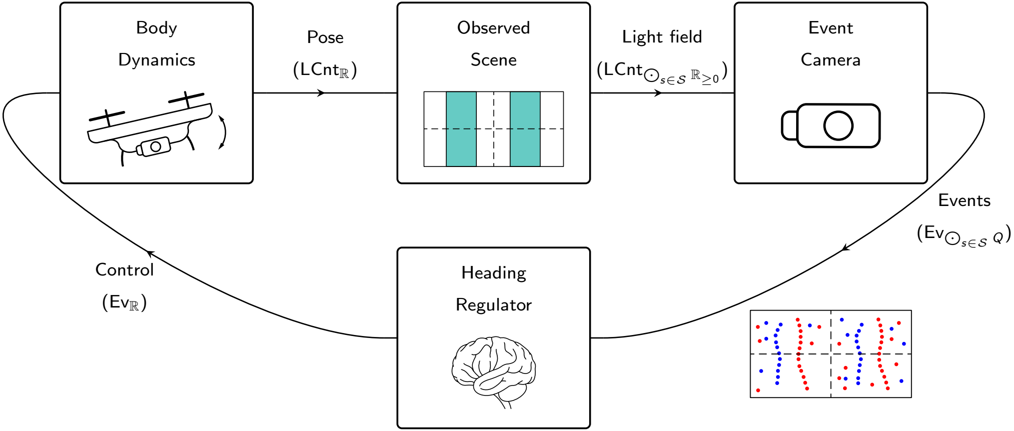

Outlook

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- We developed a framework which allows putting together different components (events, clocks, continuous-time).
- This framework is *descriptive*.

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- This framework is *descriptive*.

We now want to work on the *synthesis* of such components:

- The theory of *co-design* allows design across field boundaries.